Eur. Phys. J. B **46**, 409–417 (2005)
DOI: 10.1140/epjb/e2005-00269-1

THE EUROPEAN
PHYSICAL JOURNAL B

Anomalous sensitivity to initial conditions and entropy production in standard maps: Nonextensive approach

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Received 2 March 2005
Published online 18 August 2005 –

Abstract. We perform a throughout numerical study of the average sensitivity to initial conditions and entropy production for two symplectically coupled standard maps focusing on the control-parameter region close to regularity. Although the system is ultimately strongly chaotic (positive Lyapunov exponents), it first stays lengthily in weak-chaotic regions (zero Lyapunov exponents). We argue that the nonextensive generalization of the classical formalism is an adequate tool in order to get nontrivial information about the first stage of this crossover phenomenon. Within this context we analyze the relation between the power-law sensitivity to initial conditions and the entropy production.

PACS. 05.10.-a Computational methods in statistical physics and nonlinear dynamics – 05.45.Ac Low-dimensional chaos

1 Introduction

Hamiltonian chaotic dynamics is related to the description of irregular trajectories in phase space. This characteristic is closely linked to the instability of the system and the entropy growth (see, e.g., [1]). Chaotic dynamics is in fact associated with positive Lyapunov coefficients, which corresponds, through the Pesin equality [2], to a positive Kolmogorov-Sinai entropy rate [3]. However, many physical, biological, economical and other complex systems exhibit more intricate situations, associated to a phase space that reveals complicated patterns and anomalous, not strongly chaotic, dynamics [4]. In many of these cases, the system displays an algebraic sensitivity to initial conditions and the use of the Boltzmann-Gibbs (BG) entropic functional $S_{BG} \equiv -\sum_{i=1}^{W} p_i \ln p_i$ for the definition of quantities such as the Kolmogorov-Sinai entropy rate only provides trivial information. It has been first argued [5], then numerically exhibited [6–10], and finally analytically shown [11,12] that under these conditions the nonextensive generalization [13] of the BG entropic form adequately replaces these concepts. This generalization has been successfully checked for many cases where long-range interactions, long-range microscopic memory, some kind of fractalization of the phase space are inherent. Indeed, in the case of one-dimensional dissipative maps, following pioneering work [14] on asymptotic algebraic sensitivity to initial conditions, it has been numerically [5–10] and analytically proved [11] that the nonextensive formalism provides a meaningful description of the critical points where the Lyapunov coefficient vanishes. Moreover, through the nonextensive entropy it is possible to prove a remarkable generalization of the Pesin equality [12].

On the basis of these results, we present here a nonextensive approach to the description of complex behaviors associated to Hamiltonian systems that satisfy the Kolmogorov-Arnold-Moser (KAM) requirements (see, e.g., [15]). The phase space consists in this case of complicated mixtures of invariant KAM-tori and chaotic regions. A chaotic region is in contact with critical KAM-tori whose Lyapunov coefficients are zero, and a chaotic orbit sticks to those tori repeatedly with a power-law distribution of sticking times (see, e.g., [16] and references therein). It is known that, under these conditions, one main generic characteristic feature of Hamiltonian chaos is its *nonergodicity*, due to the existence of a finite measure of the regularity islands area. The set of islands is fractal, and thin strips near the islands' boundary, which

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are termed boundary layers, play a crucial role in the system dynamics. In addition to these processes, there are effects that arise solely because of the high enough dimensionality of the system. One of these effects is Arnold diffusion [17], that occurs if the number of degrees of freedom of the Hamiltonian system is larger than two [1,15]. In general, for the usual case of many-body short-range-interacting Hamiltonian systems, the combination of these nonlinear dynamical phenomena typically makes the system to leave the generic situation of nonergodicity and eventually yields *ergodicity*.

Along these lines, anomalous effects for the sensitivity to initial conditions and for the entropy production have already been investigated within the nonextensive formalism [18] for a paradigmatic model that exhibits the KAM structure, namely the well known standard map [19]. In the present paper we focus on two symplectically coupled standard maps. The choice of *two* coupled maps is because this is the lowest possible dimension ($d = 4$) of Hamiltonian maps where Arnold diffusion does take place, an effect that has a fundamental impact for the macroscopic effects associated with the dynamics of the system, since it guarantees an unique (connected) chaotic sea (see, e.g., [20,21]).

In reference [22] it was numerically shown that, for low-dimensional Hamiltonian (hence symplectic) maps, the Kolmogorov-Sinai entropy rate $\mathcal{K}$ coincides with the entropy K produced per unit time by the dynamical evolution of a statistical ensemble of copies of the system that is initially set far-from-equilibrium, i.e., $K = \mathcal{K}$. More precisely, considering an ensemble of N copies of the map and a coarse graining partition of the phase space composed by W nonoverlapping (hyper)cells (with nonvanishing d hypervolumes), at each iteration step t a probability distribution $\{p_i\}_{i=1,2,\ldots W}$ is defined by means of the occupation number N_i of each cell, $p_i \equiv N_i/N$ ($\sum_i p_i = 1$), and we have that

$$K \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{\langle S_{BG}\rangle(t)}{t}, \qquad (1)$$

where the average $\langle\rangle$ is taken considering the dynamical evolution of different ensembles, all starting far-from-equilibrium, but with different initial conditions. In this paper, as far-from-equilibrium initial conditions, we will consider an ensemble of N copies of the map all randomly distributed inside a *single* (hyper)cell of the partition. In this way the initial entropy is zero, since all but one probabilities vanish. Averages are then obtained by sampling the whole phase space changing the position of the initial (hyper)cell.

On the other hand, one can consider the sensitivity to initial conditions

$$\xi(\mathbf{x}(0), \delta\mathbf{x}(0), t) \equiv \lim_{||\delta\mathbf{x}(0)||\to 0} \frac{||\delta\mathbf{x}(t)||}{||\delta\mathbf{x}(0)||}, \qquad (2)$$

that in general depends on the phase space initial position $\mathbf{x}(0)$ and on the direction in the tangent space $\delta\mathbf{x}(0)$. If the system is chaotic, the exponential sensitivity to initial conditions defines a spectrum of d Lyapunov coefficients $\{\lambda^{(k)}\}_{k=1,2,\ldots d}$ coupled in pairs (d being the phase space dimension), where each element of the pair is the opposite of the other (symplectic structure). By the Pesin identity we have that $\sum_{\lambda^{(k)}>0} \lambda^{(k)} = \mathcal{K}$ and the result in [22] states then that

$$\sum_{\langle\lambda^{(k)}\rangle>0} \langle\lambda^{(k)}\rangle = K, \qquad (3)$$

where once again $\langle\rangle$ stands for average over different initial data.

Now, when the largest Lyapunov coefficient vanishes, equation (3) provides only poor information ($0 = 0$, to be precise), which is not useful for distinguishing, for example, between weak chaoticity and regularity. In order to address this issue, we generalize the previous approach in the sense of the nonextensive generalization [13] of the classical (BG) formalism. Specifically, we define the q-generalized *entropy production* (corresponding to the q-generalization of the Kolmogorov-Sinai entropy rate) as

$$K_{q_e} \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{\langle S_{q_e}\rangle(t)}{t}, \qquad (4)$$

where the nonextensive entropy S_{q_e} is defined by

$$S_{q_e} \equiv \frac{1 - \sum_{i=1}^{W} p_i^{\,q_e}}{q_e - 1} \quad (q_e \in \mathbb{R};\ S_1 = S_{BG}), \qquad (5)$$

and e stands for *entropy*. In case of equiprobability, i.e., $p_i = 1/W$ $(\forall i)$, the nonextensive entropy takes the form $S_{q_e} = \ln_{q_e} W$, where $\ln_q x \equiv (x^{1-q} - 1)/(1-q)$ (with $\ln_1 x = \ln x$) is known in the literature as the q-logarithm [23]. Notice that its inverse, the q-exponential, is given by

$$\exp_q x \equiv [1 + (1-q)x]^{(1/(1-q))} \quad (\exp_1 x = e^x). \qquad (6)$$

In this paper we will show that, in situations where the largest Lyapunov coefficient tends to zero, a regime emerges (that we will discuss in more details later on) where the sensitivity to initial conditions takes the form of a q-exponential, namely

$$\xi(t) = [1 + (1-q_s)\lambda_{q_s} t]^{1/(1-q_s)} \equiv \exp_{q_s}(\lambda_{q_s} t), \qquad (7)$$

with a specific value of the index $q_s < 1$ and of the generalized Lyapunov coefficient λ_{q_s} (s stands for *sensitivity*). Correspondingly, a single value of $q_e < 1$ exists for which the generalized entropy (5) displays a regime of linear increase with time. Under these conditions, we study the relation between q_s and q_e. We will see that, differently to what happens in the case of strong chaos (where $q_e = q_s = 1$), the two indexes q_e and q_s do not coincide in general. We discuss the origin of this fact.

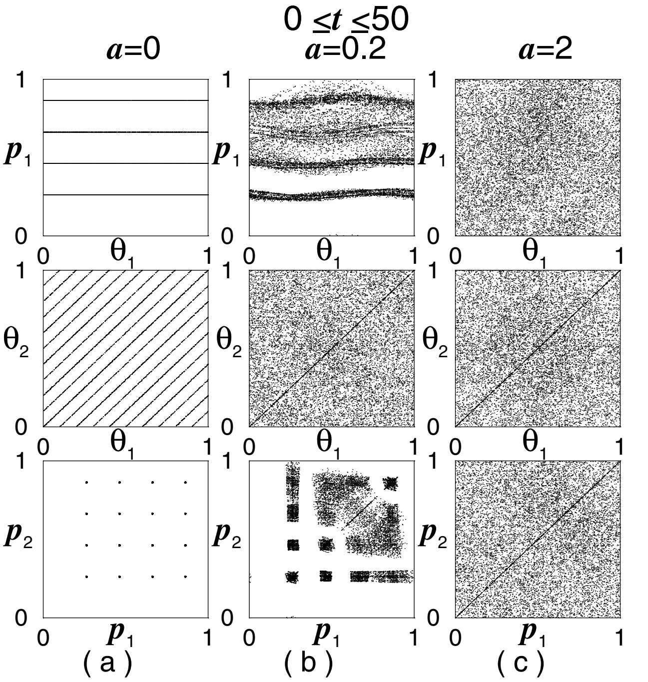

Fig. 1. Phase portrait of two symplectically coupled standard maps, equation (8), with $b = 0.5$ and $a_1 = a_2 \equiv a$. Points represent the projection of trajectories on different planes. We trace an ensemble of $N = 4^4$ points, with a uniform initial distribution, for $0 \leq t \leq 50$. (a) $a = 0$: Integrability; (b) $a = 0.2$: Weak chaos; (c) $a = 2$: Strong chaos.

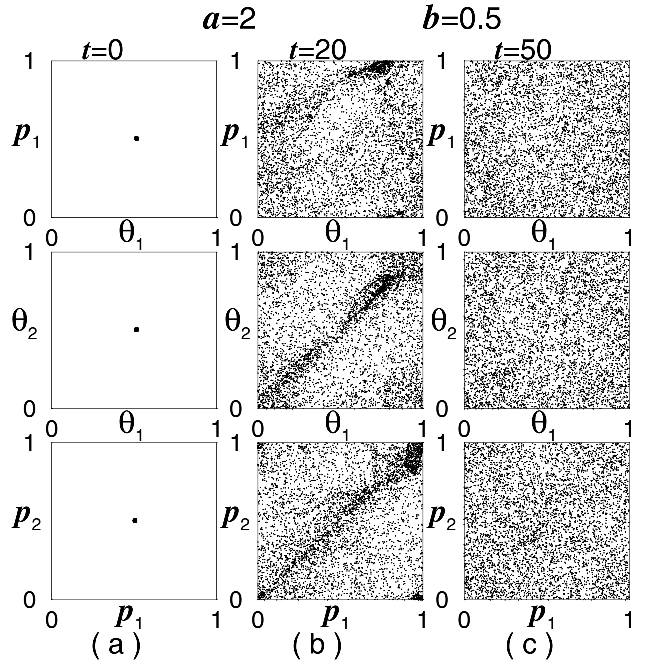

Fig. 2. Fixed time evolution of an initially out-of-equilibrium ensemble of $N = 5 \times 10^3$ copies of the map (8) for the strongly chaotic case $a = 2$, $b = 0.5$. (a) $t = 0$; (b) $t = 20$; (c) $t = 50$.

2 Sensitivity to initial conditions and entropy production

We consider a dynamical system with an evolution law given by two *symplectically* coupled standard maps:

$$\begin{aligned}
\theta_1(t+1) &= p_1(t+1) + \theta_1(t) + b\, p_2(t+1) \quad &(\text{mod } 1),\\
p_1(t+1) &= p_1(t) + (a_1/2\pi)\sin[2\pi\theta_1(t)] \quad &(\text{mod } 1),\\
\theta_2(t+1) &= p_2(t+1) + \theta_2(t) + b\, p_1(t+1) \quad &(\text{mod } 1),\\
p_2(t+1) &= p_2(t) + (a_2/2\pi)\sin[2\pi\theta_2(t)] \quad &(\text{mod } 1), \quad (8)
\end{aligned}$$

where θ_i, $p_i \in \mathbb{R}$ can be regarded respectively as an angle and an angular momentum coordinate, $a_1, a_2, b \in \mathbb{R}$, and $t = 0, 1, 2, ...$. Since the system is symplectic $\forall(a, b)$, it immediately follows that it also is *conservative*, i.e., the Jacobian determinant of the map is one ($det\{\partial[\theta_1(t+1), p_1(t+1), \theta_2(t+1), p_2(t+1)]/\partial[\theta_1(t), p_1(t), \theta_2(t), p_2(t)]\} = 1$). For the sake of definiteness, most part of this paper will deal with the setting $a_1 = a_2 \equiv a$, so that the analysis will be restricted to the case where the system is symmetric with respect to exchange $1 \leftrightarrow 2$. If the coupling parameter b vanishes, the two standard maps decouple. For a generic value of b, the system is integrable for $a = 0$, while chaoticity rapidly increases with $|a|$. For intermediate values of $|a|$, trajectories in phase space define complex structures. The phase space is non-uniform and consists of domains of chaotic dynamics (stochastic seas, stochastic layers, stochastic webs, etc.) and islands of regular quasi-periodic dynamics [15]. To illustrate these behaviors, in Figure 1a–c we show the *projection* on various planes of the dynamical evolution of an ensemble of $N = 4^4$ points uniformly distributed in phase space at $t = 0$, for $a = 2$ (strong chaos), $a = 0.2$ (weak chaos), and $a = 0$ (integrability). Reminding the fact that the shadow of a (fractal-like) sponge on a wall has a regular geometry, the presence of structures as those that appear for $a = 0.2$ in the projection of the ensemble over coordinates planes is a strong hint that even more intricate structures are present in the phase space itself. In order to further characterize the strongly and the weakly chaotic situations, Figures 2 and 3 display, for different *fixed* times, the projection of the evolution of an initially out-of-equilibrium ensemble, respectively for $a = 2$ and $a = 0.2$. In all previous figures we have fixed $b = 0.5$.

Chaotic regions produce exponential separation of initially close trajectories. Before entering in the details of our analysis, let us consider how the usual, strongly chaotic case is described inside the nonextensive formalism, both for the sensitivity to initial conditions and the entropy production. To extract the mean value of the largest Lyapunov coefficient and the mean entropy production rate, we analyze respectively the average values $\langle \ln_q \xi \rangle(t)$ and $\langle S_q \rangle(t)$, over different initial conditions. Figure 4, when $q = 1$, reproduces the results in [22] for $a1 = 3$, $a_2 = 1$ and $b = 0.5$ (strong chaos). Only for $q = 1$ both the average logarithm of the sensitivity to initial conditions and the average entropy display a regime with a *linear* increase with time, before a saturation effect due to the finiteness of the total number W of (hyper)cells. If $q < 1$ ($q > 1$), the $W \to \infty$ curve is convex (concave) for both

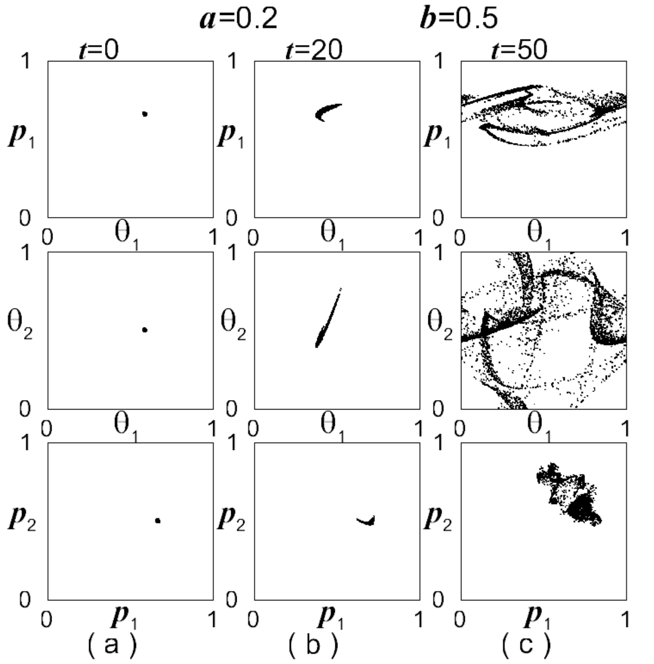

Fig. 3. Fixed time evolution of an initially out-of-equilibrium ensemble of $N = 5 \times 10^3$ copies of the map (8) for the weakly chaotic case $a = 0.2$, $b = 0.5$. (a) $t = 0$; (b) $t = 20$; (c) $t = 50$.

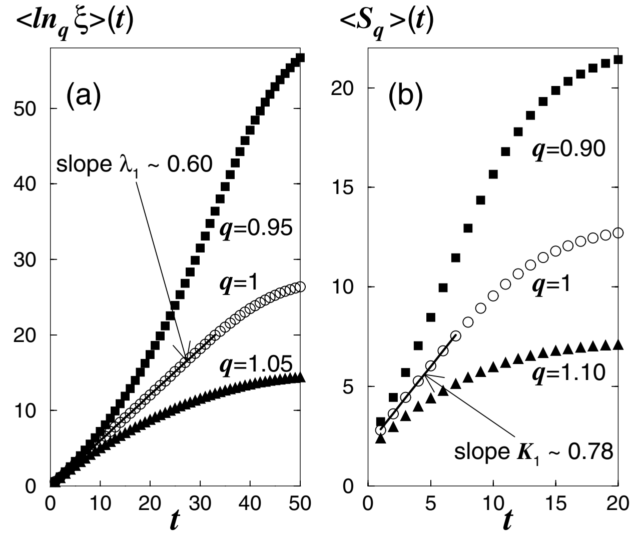

Fig. 4. Average value of the logarithm of the sensitivity to initial conditions (a) and average entropy production (b) for the map equation (8) in a strong chaos regime ($a_1 = 3$, $a_2 = 1$ and $b = 0.5$). The sensitivity to initial conditions is calculated considering the time evolution of the separation of 10^5 couple of initial data randomly distributed in the whole phase space with an initial separation of order $||\delta \mathbf{x}(0)|| \simeq 10^{-12}$ (we have checked here and elsewhere that this initial discrepancy is small enough in order to not introduce spurious numerical effects at the scales at which we are exhibiting our results). The entropy is calculated dividing the phase space in $W = 30^4$ hypercells and analyzing the spreading of an ensemble of $N = 6 \times 10^6$ data initially distributed inside a single hypercell. The calculation was repeated 2000 times, choosing the position of the initial cell at random in the whole phase space.

quantities. Notice that the slope of the linear increase of $\langle \ln \xi \rangle(t)$ is smaller than the slope of $\langle S_{BG} \rangle(t)$, as, since the phase space is 4-dimensional, there are *two* positive Lyapunov exponents to be considered in equation (3).

We turn now our attention to the weak chaos. Figure 5 shows that, fixing $b = 0.5$ and decreasing now $a_1 = a_2 \equiv a$, the mean value of the largest Lyapunov coefficient tends to zero as a consequence of the fact that dynamics is now more constrained by structures in phase space. Particularly, for $a \lesssim 0.8$ an initial phase emerges for which $\langle \ln \xi \rangle(t)$ is not linear (see the inset of Fig. 5). We refer to this phase as the *weakly chaotic regime*. As a tends to zero, the crossover time τ (see its definition in Fig. 6) between this initial regime and the one characterized by a linear increase of $\langle \ln \xi \rangle(t)$ increases, so that the weakly chaotic phase becomes more and more important. The sensitivity to initial conditions in correspondence to this initial regime is power-law. In fact, Figure 7 exhibits that, for specific values $q_s(a) < 1$, $\langle \ln_{q_s} \xi \rangle(t)$ grows *linearly* during this regime and starts later to be convex when the sensitivity becomes exponential, after the crossover. For a approaching zero, the crossover time between this initial regime and the strongly chaotic regime (characterized by a linear increase of $\langle \ln \xi \rangle(t)$) *diverges* (see inset of Fig. 7). Note that q_s *depends* on a. For integrability we have, as expected, $q_s(0) = 0$.

The weakly chaotic regime can also be detected by an analysis of the entropy production, performed as described in Section 1. In Figure 8 we take as an example the case $b = 0.5$ and $a = 0.2$. For the weakly chaotic regime the result is similar to Figure 4b, but the value for which the entropy grows linearly is now $q_e \simeq 0.6$ instead of $q_e = 1$.

Figure 9 synthesizes the behavior of q_s and q_e for two different fixed coupling constants, namely $b = 0.5$ and $b = 0.2$. At variance with what happens in the strongly chaotic regime, in the weakly chaotic one q_s and q_e are *not* the same. In the next section we discuss the origin of this difference.

Another numerical result is that, while the the generalized Lyapunov coefficient appears to be almost independent from a ($\lambda_{q_s} = 0.57$ and $\lambda_{q_s} = 0.50$ for $b = 0.20$ and $b = 0.50$ respectively), the generalized Kolmogorov-Sinai entropy rate K_{q_e} exhibits, within our numerical precision, a slight decrease as a increases. We do not have yet a clear explanation for this observation.

3 Discussion

To understand the origin of the discrepancy, in the weakly chaotic regime, between q_s and q_e some simple geometric considerations are in order.

Firstly, we notice the following property of the q-logarithmic function:

$$\ln_q W^\alpha = \alpha \ln_{q'} W \, , \tag{9}$$

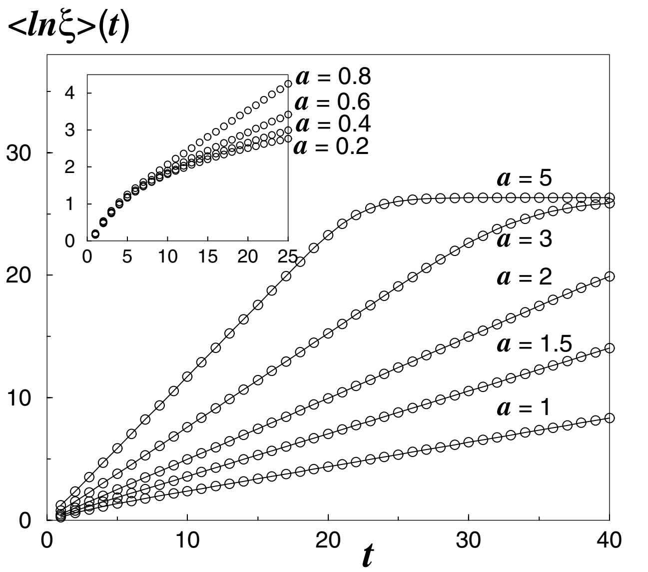

Fig. 5. Average value of the logarithm of the sensitivity to initial conditions for the map equation (8) with $b = 0.5$ and various values of a. The calculation is the result of the analysis of 10^5 couple of initial data randomly distributed in the whole phase space with an initial separation of order $||\delta\mathbf{x}(0)|| \simeq 10^{-12}$. The inset reproduces the same axes of the large figure, for small values of the ordinate. The solid lines are guides to the eye. Notice that the slopes of the straight portions monotonically decrease from above unity for $a = 5$ to about 0.05 for $a = 0.2$, thus approaching zero in the limit $a \to 0$.

where $\alpha \in \mathbb{R}$ and q, q' are related by

$$1 - q = \frac{1 - q'}{\alpha}. \tag{10}$$

Notice that, for all $\alpha > 1$, $q' = 1$ implies $q = 1$, whereas $q' < 1$ implies $q' < q < 1$.

Secondly, for simplicity, we consider the case of a 2-dimensional phase space where the dynamical evolution is *symmetric* with respect to the exchange of the coordinates. Let us call $W^{(2)} = W^{(1)} \otimes W^{(1)}$ the total number of bidimensional cells composed by a partition of $W^{(1)}$ intervals in each coordinate. We suppose to set a far-from-equilibrium initial ensemble inside a single 2-dimensional cell and that the dynamical evolution of each trajectory in the ensemble implies a spreading of the ensemble in phase space. Let $W^{(1)}(t)$ and $W^{(2)}(t)$ be respectively the number of occupied 1-dimensional intervals in one of the two coordinate axes and the number of occupied 2-dimensional cells, at time t ($W^{(1)}(0) = W^{(2)}(0) = 1$). The relation between $W^{(1)}(t)$ and $W^{(2)}(t)$ depends on the details of the dynamical evolution. Two limiting cases are

$$W^{(2)}(t) \propto W^{(1)}(t) \quad \text{and} \quad W^{(2)}(t) \propto [W^{(1)}(t)]^2. \tag{11}$$

The former is realized for instance when there is a predominant direction along which the ensemble stretches, so that the dynamical evolution of the ensemble produces

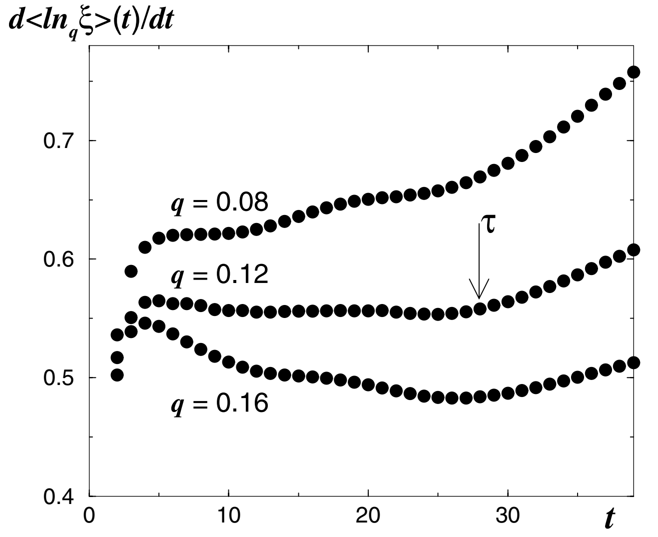

Fig. 6. Time dependence of the time derivative of the average q-logarithmic sensitivity for $a = 0.05$ and $b = 0.5$ for typical values of q. We verify that $q_s \simeq 0.12$ and $\lambda_{q_s} \simeq 0.56$. The arrow indicates the value of τ.

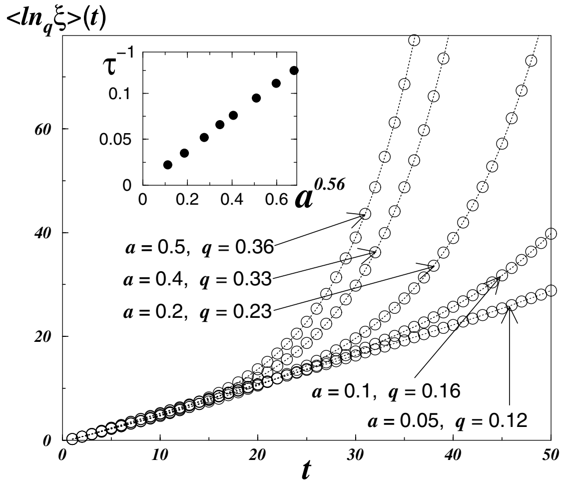

Fig. 7. Average value of the q_s-logarithm of the sensitivity to initial conditions for the map equation (8) with $b = 0.5$ and various values of a. The calculation is the result of the analysis of 10^8 couple of initial data randomly distributed in the whole phase space with an initial separation of order $||\delta\mathbf{x}(0)|| \simeq 10^{-12}$. q_s changes with a (see also Fig. 9) and it was determined by a constant derivative condition during the weakly chaotic regime. The dotted lines are guides to the eye. In the inset, inverse of the crossover time vs. $a^{0.56}$.

filaments in the 2-dimensional phase space and it is essentially unidimensional. The latter happens for example when the dynamical evolution in the two coordinates is decoupled, so that $W^{(2)}(t)$ is simply the Cartesian product $W^{(1)}(t) \otimes W^{(1)}(t)$. Now, if we suppose that $\ln_{q_s} W^{(1)}(t) \propto t$, for a specific value of the parameter q_s (we are using the notation q_s consistently with the fact that ξ represents the growth of a unidimensional arc), by means of equations (9) and (10) we have that $\ln_{q_e} W^{(2)}(t) \propto t$ for

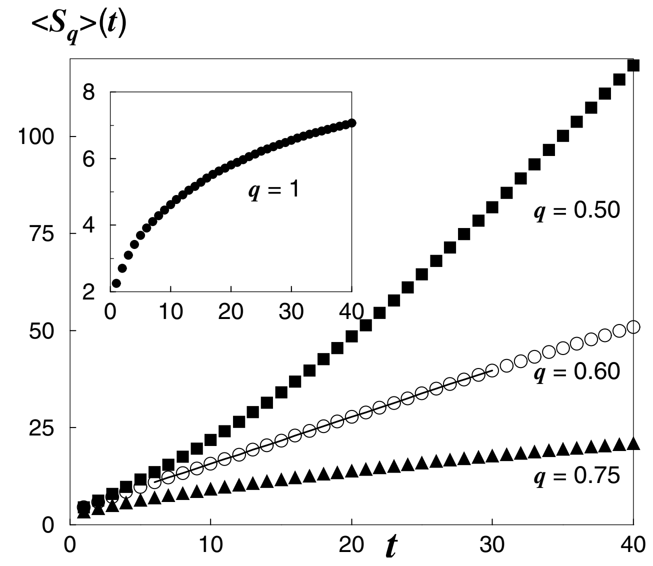

Fig. 8. Average entropy production for the map equation (8) in a weak chaos regime $b = 0.5$ and $a = 0.2$. Phase space was divided into $W = 30^4$ hypercells and the spreading of an ensemble of $N = 6 \times 10^6$ initial data was analyzed. The calculation was repeated 3000 times, choosing the position of the initial cell at random in the whole phase space. We verify that $q_e \simeq 0.6$ and $K_{q_e} \simeq 1.2$; q_e changes with a (see also Fig. 9) and it was determined by a constant derivative condition during the weakly chaotic regime. The straight line is the result of a linear interpolation.

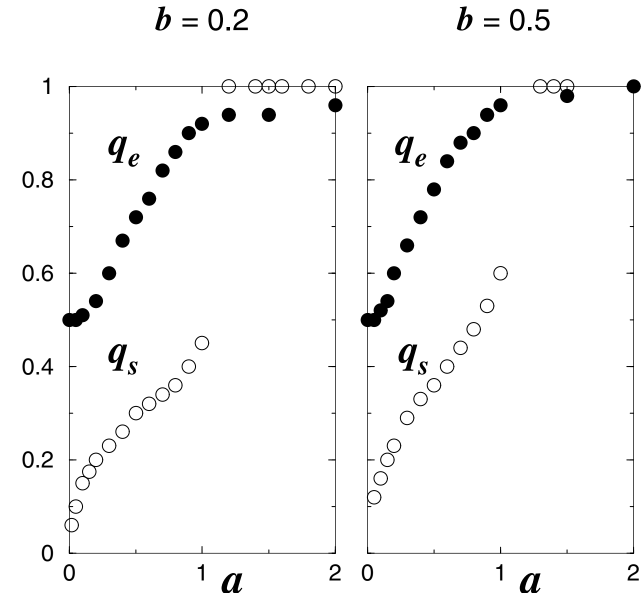

Fig. 9. Behavior of q_s and q_e as a function of a, for $b = 0.2$ e $b = 0.5$. The two parameters coincide only in the case of strong chaos ($|a| \gg 1$), when $q_s = q_e = 1$.

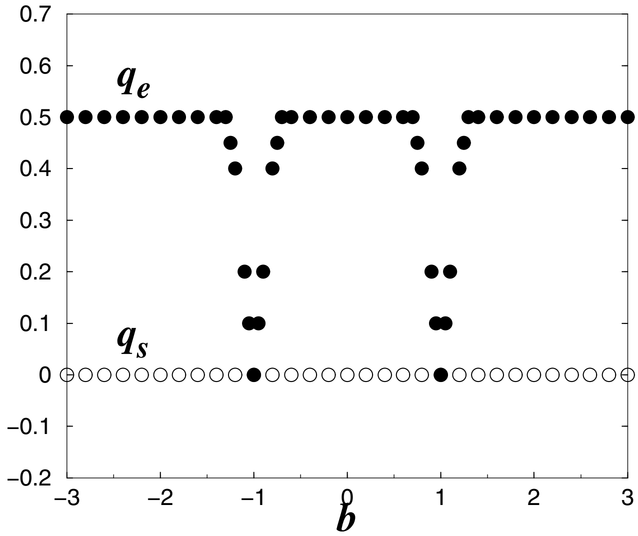

Fig. 10. Behavior of q_s and q_e as a function of b, for the integrable case $a = 0$. The two indexes coincide only for the special condition $b = \pm 1$ (see text for details), when $q_s = q_e = 0$. For almost all other values of b we verify $1 - q_e = (1 - q_s)/2$.

the values

$$q_e = q_s \quad \text{and} \quad q_e = 1 - \frac{1 - q_s}{2}, \tag{12}$$

respectively, for the two previous limiting cases (in Eq. (11)). Notice that $q_s = 1 \Rightarrow q_e = 1$ for both cases. Of course, equation (12) can be straightforwardly generalized for arbitrary dimensions of the phase space.

This geometrical analysis applies to the problem of the entropy production, strictly speaking, only if the ensemble evolves according to a uniform distribution in phase space (equiprobability), so that the value of the generalized entropy is given by $\ln_q W$. Nonetheless, we will see now that the same analysis is useful to understand the dynamical behavior of the model that we are studying. We start by considering the effect of the coupling term b in the special case $a = 0$, that corresponds to integrability. Figure 10 represents the behavior of $q_s(b)$ and $q_e(b)$ for this case. As expected, q_s is zero, thus displaying that the sensitivity to initial conditions increases linearly. On the other hand, the behavior of q_e is understood if we analyze in some details the dynamics. We have that p_1 and p_2 are just conserved along any trajectory and all the interesting dynamical evolution take place in the (θ_1, θ_2)-plane by means of the iteration laws

$$\begin{aligned} \theta_1(t+1) &= p_1(0) + \theta_1(t) + b\, p_2(0), \\ \theta_2(t+1) &= p_2(0) + \theta_2(t) + b\, p_1(0). \end{aligned} \tag{13}$$

As it is apparent, the two coordinates are decoupled and for $b \neq \pm 1$ the growth of the ensemble is 2-dimensional as confirmed by Figure 11 (first row). Under these conditions, we have $W^{(4)}(t) \propto W^{(2)}(t) \propto [W^{(1)}(t)]^2$, so that, applying the previous analysis, we obtain $q_e = 0.5$. For $b = \pm 1$ the maps further degenerates, since now we have one more conserved quantity, i.e.,

$$\theta_1(t+1) \mp \theta_2(t+1) = \theta_1(t) \mp \theta_2(t). \tag{14}$$

This implies $W^{(4)}(t) \propto W^{(2)}(t) \propto W^{(1)}(t)$ (see Fig. 11, second row), and we obtain $q_e = q_s = 0$. Around $b = \pm 1$ there is then a transition from $q_e = 0.5$ to $q_e = 0$, as it is exhibited in Figure 10.

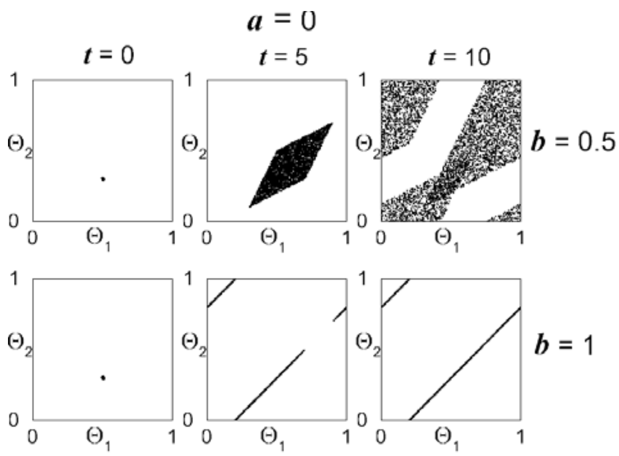

Fig. 11. Fixed time evolution of an initially out-of-equilibrium ensemble of $N = 5\times10^3$ copies of the map (8) for the integrable case $a = 0$, $b = 0.5$ (first row) and $b = 0.2$ (second row). We display the projection of the ensemble on the plane (θ_1, θ_2).

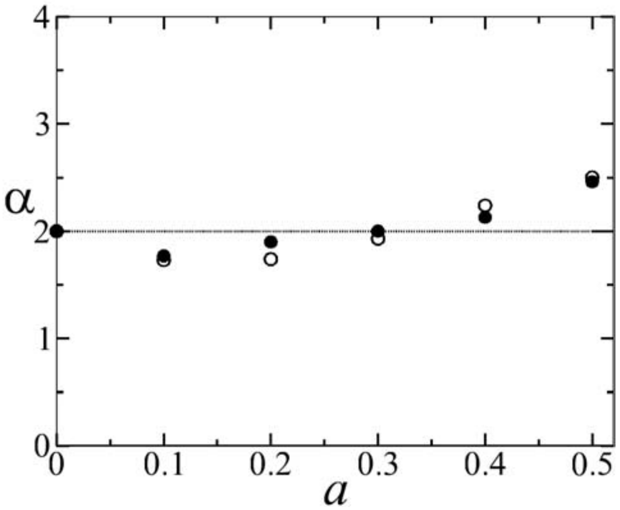

Fig. 12. Behavior of α as a function of a, for $b = 0.2$ (open circles) and $b = 0.5$ (full circles). For a large part of the weakly chaotic region α is almost constant and equal to 2. The line is a guide to the eye. We do not show the data for $a > 0.5$ because the time interval within which we define q_s and q_e is not large enough to provide a confident value of these quantities.

We can consider now the generic weakly chaotic case ($0 < |a| \ll 1$). In Figure 12 we estimate the (below defined) factor α that connects the 4-dimensional analysis performed by the entropy to the one-dimensional inspection provided by the sensitivity to initial conditions, by means of the relation (see Eq. (10))

$$\alpha \equiv \frac{1-q_s}{1-q_e}. \tag{15}$$

For both $b = 0.5$ and $b = 0.2$, small positive values of a yield $\alpha \simeq 2$, as it is for the integrable case $a = 0$. This exhibits an essentially 2-dimensional growth of the portion of the phase space occupied by the ensemble.

At this point we can advance a conjecture for the more general case of a phase space that is not symmetric under the interchange of coordinates. This conjecture will allow for a better understanding of the reason of the discrepancy between q_s and q_e. In the case of strong chaos in a symplectic nonlinear system, the growth of the hypervolume containing the initially out-of-equilibrium ensemble is essentially $d/2$-dimensional (d being the phase space dimension of the system). In fact, if $\{\lambda^{(k)}\}_{k=1,2,\ldots,d/2}$ with $\lambda^{(k)} > 0\ \forall k$ is the spectrum of *positive* Lyapunov coefficients, and we call $\xi^{(k)}$ the sensitivity to initial conditions associated to the direction defining $\lambda^{(k)}$, we have

$$W^{(d)}(t) \propto W^{(d/2)}(t) \tag{16}$$

$$\propto \xi^{(1)}(t)\,\xi^{(2)}(t)\,\cdots\,\xi^{(d/2)}(t) \tag{17}$$

$$\propto \exp(\lambda^{(1)}t)\ \exp(\lambda^{(2)}t)\ \cdots\ \exp(\lambda^{(d/2)}t) \tag{18}$$

$$\propto \exp\left(\sum_{k=1}^{d/2}\lambda^{(k)}t\right). \tag{19}$$

We remind that $W^{(d)}(t)$ and $W^{(d/2)}(t)$ in equation (16) respectively are the numbers of occupied d-dimensional and $d/2$-dimensional hypercells. Taking the logarithm of relation (19) we obtain essentially the Pesin equality, namely

$$K = \sum_{k=1}^{d/2}\lambda^{(k)}, \tag{20}$$

where $\ln W^{(d)}(t) \simeq Kt$.

If we now assume that the sensitivities to initial conditions along the various directions of the expansion are $q^{(k)}$-exponential power-laws (which implies that the largest Lyapunov coefficient vanishes)

$$\xi^{(k)} \propto t^{1/(1-q^{(k)})}, \tag{21}$$

the previous relations (16–19) transform, for very long times, into

$$W^{(d)}(t) \propto W^{(d/2)}(t) \tag{22}$$

$$\propto \xi^{(1)}(t)\,\xi^{(2)}(t)\,\cdots\,\xi^{(d/2)}(t) \tag{23}$$

$$\propto t^{1/(1-q^{(1)})}\ t^{1/(1-q^{(2)})}\ \cdots\ t^{1/(1-q^{(d/2)})} \tag{24}$$

$$\propto t^{\sum_{k=1}^{d/2} 1/(1-q^{(k)})}, \tag{25}$$

and, since $\ln_{q_e} W^{(d)}(t)$ is linear with time for q_e (i.e., $S_{q_e} \propto [W^{(d)}(t)]^{1-q_e} \propto t$, hence $W^{(d)}(t) \propto t^{1/(1-q_e)}$) we obtain

$$\frac{1}{1-q_e} = \sum_{k=1}^{d/2}\frac{1}{1-q^{(k)}}. \tag{26}$$

Note that, for $q^{(k)} < 1\ \forall k$, we have $q_e > q^{(k)}; \forall k$. Note also that for $q^{(k)} = q_s\ \forall k$ equation (26) reduces to equation (15) identifying $\alpha \equiv d/2$.

Since equation (26) relates an information associated with the entropy production to an information associated with the sensitivity to initial conditions, it plays, for weakly chaotic dynamical systems, the role played by the Pesin equality for strongly chaotic ones. Working out a generalization of traditional methods for the calculation of the complete spectrum of the Lyapunov coefficients (e.g., [24]), it might be possible to numerically verify the validity of equation (26) [25].

Equations (20) and (26) can be unified as follows ($t \gg 1$)

$$\exp_{q_e}(K_{q_e} t) \simeq \prod_{k=1}^{d/2} \left[\exp_{q^{(k)}}(\lambda^{(k)}_{q^{(k)}}\ t)\right]. \qquad (27)$$

This relation hopefully is, for large classes of dynamical systems yet to be qualified, a correct conjecture. If so it is, then it certainly constitutes a powerful relation. Indeed, let us summarize some of its consequences:

(i) if the system is strongly chaotic, i.e., if it has positive Lyapunov exponents, we have $q_e = q_s = 1$ and equation (20) holds;

(ii) if the system is weakly chaotic (hence its largest Lyapunov exponent vanishes), we have that equation (26) holds;

(iii) if the system is weakly chaotic and there is only *one* dimension within which there is mixing, then, not only $q_e = q_s$, but also

$$K_{q_e} = \lambda_{q_s}\ , \qquad (28)$$

as already known for unimodal maps such as the logistic one [11,12];

(iv) if the system is weakly chaotic and there is isotropy in the sense that $q^{(k)} = q_s$ ($\forall k = 1, ..., d/2$), we have that

$$\frac{1}{1-q_e} = \frac{d/2}{1-q_s}\ . \qquad (29)$$

The special case $q_s = 0$ yields

$$q_e = 1 - \frac{2}{d}\ . \qquad (30)$$

It is suggestive to notice that, discussing a Boltzmann transport equation concerning a fluid model with Galilean-invariant Navier-Stokes equations in a d_{NS}-dimensional Bravais lattice, Boghosian et al. [26] obtained $q_e = 1 - 2/d_{NS}$, and that, for a lattice Lotka-Volterra d_{LV}-growth model, Tsekouras et al. [27] obtained $q_e = 1 - 1/d_{LV}$ (NS and LV stand respectively for *Navier-Stokes* and *Lotka-Volterra*). Moreover, using the duality relation $q' = 2-q$ which connects entropic indexes smaller than one with those larger than one (see, e.g., [12]), one obtains $q'_e = 1 + 2/d$, and this is precisely what occurs for Hamiltonian systems with temperature fluctuations (see, e.g., Eq. (29) in [31])[1].

[1] We are grateful to a Referee for pointing out this interesting connection.

4 Conclusions

We have discussed the (average) sensitivity to initial conditions and entropy production for a 4-dimensional dynamical system composed by two symplectically and symmetrically coupled standard maps, focusing on phase space configurations characterized by the presence of complex (fractal-like) structures. Under these conditions, coherently with previous 2-dimensional observations [18], we have detected the emergence of long-lasting regimes characterized by a power-law sensitivity to initial conditions, whose duration diverges when the map parameters tend to the values corresponding to integrability. While the classical BG formalism characterizes these anomalous regimes only trivially by means of the Pesin equality [2] (we have $K = 0$ and $\lambda^{(k)} = 0\ \forall k$ in equation (3)), the nonextensive formalism [13], through a generalization of the Pesin equality (see also [5,12]), provides a meaningful nontrivial description for these regimes. Specifically, we have shown that, during these anomalous regimes (here called weakly chaotic) the average sensitivity to initial conditions is a q_s-exponential power-law (6), with $q_s < 1$, and that the corresponding entropy production is (asymptotically) *finite* only for a generalized q_e-entropy (5), with $q_e < 1$. We have discussed the relation between q_s and q_e, both numerically and analytically and we propose an appealing generalization of the Pesin equality, namely equation (27). By means of the difference between q_s and q_e we obtain useful information about the dimensionality which is associated to the spreading of the initially out-of-equilibrium dynamical ensemble. If this dimensionality is equal to one, we have the result $q_e = q_s$ and $K_{q_e} = \lambda_{q_s}$, as already conjectured [5] and proved [12] for the edge of chaos of unimodal maps.

The present results concern low-dimensional conservative (Hamiltonian) maps, but the scenario we have described here may serve as well for the discussion of analogous effects arising when many maps are symplectically coupled (see, e.g., [21]), and even for the case of many-body interacting Hamiltonian systems. Short-range interactions would typically yield strong chaos, and long-range interactions may typically lead to weak chaos (consistently with the results in [28,29]). In fact, for isolated many-body long-range-interacting classical Hamiltonians there are vast classes of initial conditions for which metastable (or quasi-stationary) states are currently observed, later on followed by a crossover to the usual Boltzmann-Gibbs thermal equilibrium (see, for instance, [30]). The duration of the metastable state *diverges* with the number N of particles in the system, in such a way that the limits $N \to \infty$ and $t \to \infty$ do *not* commute. Since it is known that such systems may have vanishing Lyapunov spectrum, it is allowed to suspect that the scenario in the metastable state is similar to the weakly chaotic one described in the present paper, whereas the crossover to the BG equilibrium corresponds, in the system focused on here, to the crossover to the $q_e = q_s = 1$, strongly chaotic regime. These and other crucial aspects can in principle be verified, for instance, on systems with very many (and not only two, as here) symplectically coupled standard maps. They would provide insightful information about macroscopic systems and their possible universality classes of nonextensivity (characterized by index(es) q). Efforts along this line are surely welcome.

We acknowledge useful discussions with C. Anteneodo, E.P. Borges, L.G. Moyano, A. Rapisarda and A. Robledo, as well as partial financial support by Pronex/MCT, Faperj, Capes and CNPq (Brazilian agencies).

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